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MOTION OF A SYMMETRIC RIGID BODY SUBJECTED TO A SPECIALIZED BODY-FIXED FORCE

T. R. KANE† and B. S. CHIA‡

Department of Applied Mechanics, Stanford University, Stanford, California

Abstract—Exact dynamical and kinematical equations governing the attitude and translational motions of a symmetric rigid body under the action of a specialized body-fixed force are formulated. From these equations, an analytical, but approximate, description of the behavior of the system is obtained and the validity of the solution is then tested by comparing its predictions with those of digital computer solutions of the exact equations of motion.

INTRODUCTION

THE problem of determining the motion of a rigid body under the action of a constant, body-fixed force whose line of action fails to pass through the center of mass is so simple from a physical point of view that it may be surprising that no complete, analytical solution appears to be possible. In fact, not even the Euler dynamical equations for this problem have been solved in closed form for the general case, that is, for a body with unequal centroidal principal moments of inertia and for an arbitrary orientation of the line of action of the force. (Leimanis and Lee [1, 2] developed expressions for the angular velocity for the special case of a force placed so that its moment about the center of mass is parallel to a centroidal principal axis, but these expressions involve integrals which, in general, cannot be evaluated by quadrature; and Grammel [3] constructed approximate solutions of Euler's dynamical equations.) Hence it is natural to seek solutions of related, but somewhat simpler problems.

A major step in the direction of decreased analytical complexity can be taken by confining attention to a body whose inertia ellipsoid for the mass center is axisymmetric. One of the first to deal with this problem was Bödewadt [4], who showed that the solution of Euler's dynamical equations can be expressed in terms of integrals of the Fresnel type.§ Unfortunately, the next step of his solution, that of integrating a set of kinematical equations, was based on a false premise, because, as may be verified by substitution from equation (49) into (42) of [4], the method proposed by Bödewadt is valid only when the matrices U and W' commute, which is not the case for the problem at hand. Consequently, the orientation of the body remains to be determined as a function of time. The same problem was also attacked by Auelmann [5], who attempted to find a closed-form solution to the kinematical equations. While falling short of this goal, he succeeded in obtaining an expression for the angle between the symmetry axis and the initial direction of the symmetry axis for the special case when the initial angular velocity is normal to the symmetry axis.

† Professor of Applied Mechanics.
‡ Research Assistant.
§ An English account of Bödewadt's work appears in chapter 10 of Leimanis' book [1].

A further simplification results from restricting oneself to consideration of body-fixed forces whose lines of action lie in the plane containing the mass center and normal to the symmetry axis. A very elegant treatment of one facet of this problem was presented by Valeev [6], who used continued fractions to obtain a description of the motion of a space-fixed unit vector relative to three body-fixed unit vectors parallel to principal axes of inertia for the mass center. It is the purpose of the present paper to present a complete, albeit approximate, solution of the same problem, for both the rotational *and* translational motion.

The paper begins with a detailed description of the system to be studied and the mathematical quantities to be used in the analysis are defined. Next, exact dynamical and kinematical equations governing attitude motions are formulated; the dynamical equations are solved exactly; and an approximate solution to the kinematical equations is obtained. The validity of the approximate solution is then tested by comparison with digital computer integrations of the exact equations. Finally, the motion of the mass center is studied in terms of an approximate analytical solution and results are once again tested by means of comparisons with solutions obtained by numerical integrations.

SYSTEM DESCRIPTION

The system to be analyzed is shown in Fig. 1, where B represents an axially symmetric rigid body of mass m. The center of mass of B, designated B^*, is located by a position vector $\mathbf{X}$ relative to a point 0 that is fixed in an inertial reference frame A. Mutually perpendicular unit vectors $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2$ are fixed in A and mutually perpendicular unit vectors $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2$ are fixed in B parallel to principal axes of inertia of B for B^*, with $\mathbf{b}_3$ parallel to the symmetry axis of B.

The moments of inertia of B with respect to lines passing through B^* and parallel to $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_3$ have the values I, I and J, respectively.

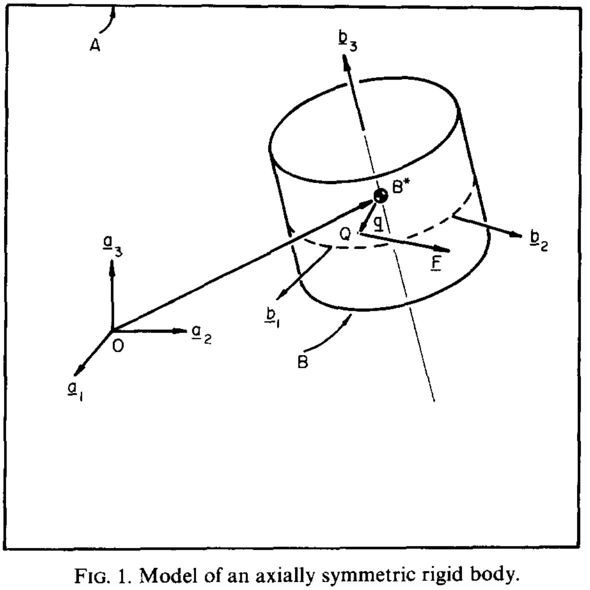

FIG. 1. Model of an axially symmetric rigid body.

It is presumed that B is subjected to the action of a force $\mathbf{F}$ of constant magnitude and fixed direction in B. This force is applied to B at a fixed point Q which is located relative to B^* by a vector $\mathbf{q}$. Furthermore, it is assumed that both $\mathbf{F}$ and $\mathbf{q}$ are perpendicular to $\mathbf{b}_3$. Hence, if the nine scalar quantities x_i, F_i and $q_i (i = 1, 2, 3)$ are defined as

$$x_i \triangleq \mathbf{X} \cdot \mathbf{a}_i \tag{1}$$

$$F_i \triangleq \mathbf{F} \cdot \mathbf{b}_i \qquad (i = 1, 2, 3) \tag{2}$$

$$q_i \triangleq \mathbf{q} \cdot \mathbf{b}_i \tag{3}$$

then $F_3 = q_3 = 0$.

When the force F exerted on B at Q is replaced with a force applied at B^* together with a couple of torque $\mathbf{T}$, then

$$\mathbf{T} = \mathbf{q} \times \mathbf{F} = T\mathbf{b}_3 \tag{4}$$

where T is defined as

$$T \triangleq q_1 F_2 - q_2 F_1. \tag{5}$$

ATTITUDE MOTION

The work that follows is facilitated by introducing a reference frame C which is fixed neither in the body B nor in the inertial reference frame A, but is constrained to move in such a way that a unit vector $\mathbf{c}_3$, fixed in C, remains at all times equal to the unit vector $\mathbf{b}_3$, which is fixed in B. The angular velocity of B relative to C, to be denoted by ${}^C\boldsymbol{\omega}^B$, is then necessarily parallel to $\mathbf{c}_3$ and can be expressed as

$${}^C\boldsymbol{\omega}^B = s\mathbf{c}_3 \tag{6}$$

where s is a function of time t. As will be seen presently, it is the choice of s which furnishes the key to the solution of the problem at hand. However, no matter how s is chosen, the angular velocity of C relative to A, ${}^A\boldsymbol{\omega}^C$, can be expressed as

$${}^A\boldsymbol{\omega}^C = p_1\mathbf{c}_1 + p_2\mathbf{c}_2 + p_3\mathbf{c}_3 \tag{7}$$

where $p_i (i = 1, 2, 3)$ are functions of t, and $\mathbf{c}_1$ and $\mathbf{c}_2$ are unit vectors fixed in C, perpendicular to each other and such that $\mathbf{c}_1 \times \mathbf{c}_2 = \mathbf{c}_3$. It then follows that the angular velocity of B relative to A is given by

$${}^A\boldsymbol{\omega}^B = p_1\mathbf{c}_1 + p_2\mathbf{c}_2 + (p_3 + s)\mathbf{c}_3. \tag{8}$$

As $\mathbf{b}_3 = \mathbf{c}_3$, it follows from equation (4) that the moment about B^* of the force $\mathbf{F}$ can be expressed as

$$\mathbf{T} = T\mathbf{c}_3 \tag{9}$$

and, in accordance with the angular momentum principle, the quantities p_1, p_2, p_3 and s are thus related to each other as follows:

$$I\dot{p}_1 + p_2[Js - (I - J)p_3] = 0 \tag{10}$$

$$I\dot{p}_2 - p_1[Js - (I - J)p_3] = 0 \tag{11}$$

$$J(\dot{p}_3 + \dot{s}) = T. \tag{12}$$

Equations (10)–(12) contain four unknown quantities, p_1, p_2, p_3 and s. However, s was introduced solely for the purpose of facilitating the analysis and may, therefore, be chosen at will; and a choice which, indeed, simplifies the subsequent analysis is

$$s = Lp_3 \tag{13}$$

where L is defined as

$$L \triangleq \frac{I-J}{J}. \tag{14}$$

Now, if p_{i0} denotes the initial value of $p_i (i = 1, 2, 3)$ and λ is defined as

$$\lambda \triangleq \frac{T}{I} \tag{15}$$

then substitution from equation (13) into (10)–(12), followed by integration of these equations, yields

$$p_1 = p_{10}, \qquad p_2 = p_{20}, \qquad p_3 = \lambda t + p_{30}. \tag{16}$$

Furthermore, from equations (13) and (16),

$$s = L(\lambda t + p_{30}) \tag{17}$$

and the angular velocity of B in A can now be found by substitution into equation (8). However, as it is advantageous to express ${}^A\boldsymbol{\omega}^B$ in terms of $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$, rather than $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$, a brief digression will prove useful.

If $\mathbf{c}_1$ is taken to be equal to $\mathbf{b}_1$ at $t = 0$ and θ is defined as

$$\theta \triangleq \int_0^t s \, \mathrm{d}t = L(\tfrac{1}{2}\lambda t^2 + p_{30} t) \tag{18}$$

then θ is the radian measure of the angle between $\mathbf{b}_1$ and $\mathbf{c}_1$; and if $\omega_i (i = 1, 2, 3)$ is now defined as

$$\omega_i \triangleq {}^A\boldsymbol{\omega}^B \, . \, \mathbf{b}_i \qquad (i = 1, 2, 3) \tag{19}$$

it follows from equation (8) that

$$\omega_1 = p_1 \cos\theta + p_2 \sin\theta \tag{20}$$

$$\omega_2 = -p_1 \sin\theta + p_2 \cos\theta \tag{21}$$

$$\omega_3 = p_3 + s. \tag{22}$$

Hence, if ω_{i0} denotes the initial value of $\omega_i (i = 1, 2, 3)$, then

$$p_1 = \omega_{10}, \qquad p_2 = \omega_{20}, \qquad p_3 = \lambda t + \frac{\omega_{30}}{1+L} \tag{23}$$

$$s = L\left(\lambda t + \frac{\omega_{30}}{1+L}\right) \tag{24}$$

and

$$\theta = L\left(\frac{1}{2}\lambda t^2 + \frac{\omega_{30}}{1+L} t\right) \tag{25}$$

so that, substituting into equations (20)–(22), one finds

$$\omega_1 = \omega_{10}\cos\theta+\omega_{20}\sin\theta$$

$$= \sqrt{(\omega_{10}^2+\omega_{20}^2)}\cos\left[\theta+\tan^{-1}\left(-\frac{\omega_{20}}{\omega_{10}}\right)\right] \tag{26}$$

$$\omega_2 = -\omega_{10}\sin\theta+\omega_{20}\cos\theta$$

$$= \sqrt{(\omega_{10}^2+\omega_{20}^2)}\sin\left[\theta+\tan^{-1}\left(-\frac{\omega_{20}}{\omega_{10}}\right)\right] \tag{27}$$

$$\omega_3 = (1+L)\lambda t+\omega_{30}. \tag{28}$$

Equations (25)–(28) constitute a complete solution of the dynamical equations of rotational motion for the problem at hand. In order to describe the instantaneous orientation of B in A, one must integrate yet one more set of differential equations, the so-called kinematical equations.

Conceptually, the simplest way to obtain a complete description of the attitude motion of B in A would be to solve the nine first-order differential equations governing the elements of the direction cosine matrix relating the body-fixed unit vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ to the space-fixed unit vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. However, since these equations are strongly coupled and involve the time-dependent functions $\omega_i (i = 1, 2, 3)$ [see equations (26)–(28)], there appears to be little hope of carrying such a solution to a successful conclusion. Alternatively, one may take advantage of the relative simplicity of the differential equations governing the elements of the direction cosine matrix relating $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ to $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and of the fact that, once the orientation of C in A is known, the orientation of B in A can be found easily, because B performs a simple rotational motion in C and the time-dependence of the angle θ associated with this rotational motion is already known [see equation (25)].

Accordingly, one may begin by defining $a_{ij} (i, j = 1, 2, 3)$ as

$$a_{ij} \triangleq \mathbf{a}_i \cdot \mathbf{c}_j \qquad (i, j = 1, 2, 3) \tag{29}$$

and then take advantage of the fact that the first time-derivative of $\mathbf{a}_i$ in A can be expressed as

$$\frac{{}^A\mathrm{d}\mathbf{a}_i}{\mathrm{d}t} = \frac{{}^C\mathrm{d}\mathbf{a}_i}{\mathrm{d}t}+{}^A\boldsymbol{\omega}^C\times\mathbf{a}_i \qquad (i = 1, 2, 3). \tag{30}$$

Since $\mathbf{a}_i$ is fixed in A, this derivative is equal to zero. Using equation (7), one is thus led to

$$\dot{a}_{i1}\mathbf{c}_1+\dot{a}_{i2}\mathbf{c}_2+\dot{a}_{i3}\mathbf{c}_3 = (p_3a_{i2}-p_2a_{i3})\mathbf{c}_1+(p_1a_{i3}-p_3a_{i1})\mathbf{c}_2+(p_2a_{i1}-p_1a_{i2})\mathbf{c}_3 \qquad (i = 1, 2, 3). \tag{31}$$

The quantities $p_i (i = 1, 2, 3)$ appearing in equation (31) are known functions of t [see equations (23)]; furthermore, p_2 may be set equal to zero, for this implies only that

$$\omega_{20} = 0 \tag{32}$$

that is, that ${}^A\boldsymbol{\omega}^B$ is initially perpendicular to $\mathbf{b}_2$, which can always be arranged by choosing the orientations of $\mathbf{b}_1$ and $\mathbf{b}_2$ in B suitably. Consequently, substitution from equations (16) into (31), and subsequent rearrangement, results in

$$\dot{a}_{i1} - p_{30}a_{i2} = \lambda t a_{i2} \tag{33}$$

$$\dot{a}_{i2} - p_{10}a_{i3} + p_{30}a_{i1} = -\lambda t a_{i1} \qquad (i = 1, 2, 3) \tag{34}$$

$$\dot{a}_{i3} + p_{10}a_{i2} = 0. \tag{35}$$

The initial value of a_{ij}, to be denoted by $a_{ij}(0)$, depends on the initial orientation of $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ relative to $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$. If $\mathbf{c}_i = \mathbf{a}_i$ at $t = 0$, which can always be arranged by choosing $\mathbf{a}_i (i = 1, 2, 3)$ properly, then [see equations (29)]

$$a_{ij}(0) = \delta_{ij} \qquad (i, j = 1, 2, 3) \tag{36}$$

where

$$\delta_{ij} \triangleq 1 \qquad (i = j), \qquad \delta_{ij} \triangleq 0 \qquad (i \neq j). \tag{37}$$

These initial conditions also imply that $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ are respectively parallel to $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ at $t = 0$, since $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ initially coincide with $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$, respectively.

Despite their relative simplicity, equations (33)–(35) do not appear to admit of an exact solution. Hence we shall resort to the following method of successive approximations: integrate equations (33)–(35) with $\lambda = 0$, and require the solution to satisfy equation (36). Next, substitute the resulting expressions for a_{i1} and a_{i2} into the right-hand members of equations (33)–(35), integrate these equations and again require the solution to satisfy equation (36). Finally, repeat this process once more. Since the expressions for $a_{ij}(i, j = 1, 2, 3)$ obtained in this way are quite lengthy, it is advantageous to cast the associated approximate description of the attitude motion of C in A in a simpler, equivalent form. This can be accomplished by defining p_0, P_1 and P_3 as

$$p_0 \triangleq \sqrt{(p_{10}^2 + p_{30}^2)}, \qquad P_1 \triangleq p_{10}/p_0, \qquad P_3 \triangleq p_{30}/p_0 \tag{38}$$

and then introducing a dextral set of orthogonal unit vectors, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, fixed in reference frame A in such a way that $\mathbf{e}_1$ has the same directions as the initial angular velocity of C in A, $\mathbf{e}_2 = \mathbf{a}_2$ and $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. Keeping in mind that $\mathbf{c}_i = \mathbf{a}_i$ at $t = 0$, one can then express the relation between $\mathbf{e}_i$ and $\mathbf{a}_i (i = 1, 2, 3)$ as

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} P_1 & 0 & P_3 \\ 0 & 1 & 0 \\ -P_3 & 0 & P_1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} \tag{39}$$

from which it follows by reference to equations (29) that

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} \tag{40}$$

where

$$[e_{ij}] \triangleq \begin{bmatrix} P_1 a_{11} + P_3 a_{31} & P_1 a_{12} + P_3 a_{32} & P_1 a_{13} + P_3 a_{33} \\ a_{21} & a_{22} & a_{23} \\ -P_3 a_{11} + P_1 a_{31} & -P_3 a_{12} + P_1 a_{32} & -P_3 a_{13} + P_1 a_{33} \end{bmatrix} \tag{41}$$

and

$$e_{11} \approx P_1 - P_1 P_3\left(\frac{\lambda t}{p_0}\right) - \frac{1}{2}P_1(1-4P_3^2)\left(\frac{\lambda t}{p_0}\right)^2 - \frac{1}{2}P_1P_3\left(\frac{\lambda t}{p_0}\right)^2(1-\cos p_0 t)$$
$$+P_1^3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right)\sin p_0 t + P_1P_3\left(\frac{\lambda}{p_0^2}\right)\sin p_0 t - P_1\left(\frac{\lambda}{p_0^2}\right)^2(P_3^4 + P_1^5 P_3$$
$$-2P_1^3P_3^3 + 1 + P_3^2)(1-\cos p_0 t) \tag{42}$$

$$e_{12} \approx -\frac{1}{2}P_1P_3\left(\frac{\lambda t}{p_0}\right)^2\sin p_0 t - P_1(2P_3 + P_1^3 + P_3^3 - 3P_1^3P_3^2)\left(\frac{\lambda}{p_0^2}\right)^2\sin p_0 t$$
$$-P_1\left(\frac{\lambda}{p_0^2}\right)(1-\cos p_0 t) + 3P_1P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \tag{43}$$

$$e_{13} \approx P_3 + P_1^2\left(\frac{\lambda t}{p_0}\right) - 2P_1^2P_3\left(\frac{\lambda t}{p_0}\right)^2 + \frac{1}{2}P_1^2P_3\left(\frac{\lambda t}{p_0}\right)^2(1-\cos p_0 t) - P_1^2\left(\frac{\lambda}{p_0^2}\right)\sin p_0 t$$
$$+P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right)\sin p_0 t + P_1^2\left(\frac{\lambda}{p_0^2}\right)^2(P_1^3 + P_3^3 + P_3 - 3P_1^3P_3^2)(1-\cos p_0 t) \tag{44}$$

$$e_{21} \approx P_3 \sin p_0 t + P_1^2\left(\frac{\lambda t}{p_0}\right)\sin p_0 t - 2P_1^2P_3\left(\frac{\lambda t}{p_0}\right)^2\sin p_0 t + \frac{1}{2}P_3^2\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^2\cos p_0 t$$
$$+\frac{2}{3}P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^3\cos p_0 t - \frac{1}{8}P_3^3\left(\frac{\lambda}{p_0^2}\right)^{-2}\left(\frac{\lambda t}{p_0}\right)^4\sin p_0 t$$
$$-P_1^2\left(\frac{\lambda}{p_0^2}\right)(1-\cos p_0 t) + \frac{5}{2}P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)^2\sin p_0 t - \frac{7}{2}P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right)\cos p_0 t$$
$$+P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \tag{45}$$

$$e_{22} \approx \cos p_0 t - \frac{1}{2}P_3\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^2\sin p_0 t - \frac{1}{6}P_1^2\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^3\sin p_0 t$$
$$-\frac{1}{8}P_3^2\left(\frac{\lambda}{p_0^2}\right)^{-2}\left(\frac{\lambda t}{p_0}\right)^4\cos p_0 t + P_1^2\left(\frac{\lambda}{p_0^2}\right)^2 - P_1^2\left(\frac{\lambda}{p_0^2}\right)^2\cos p_0 t$$
$$-\frac{1}{2}P_1^2\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right)\sin p_0 t \tag{46}$$

$$\begin{aligned}
e_{23} \approx\; & -P_1 \sin p_0 t + P_1 P_3 \left(\frac{\lambda t}{p_0}\right) \sin p_0 t + \frac{1}{2} P_1 (1 - 4P_3^2) \left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t \\
& - \frac{1}{2} P_1 P_3 \left(\frac{\lambda}{p_0^2}\right)^{-1} \left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t - \frac{1}{6} P_1 (1 - 4P_3^2) \left(\frac{\lambda}{p_0}\right)^{-1} \left(\frac{\lambda t}{p_0}\right)^3 \cos p_0 t \\
& + \frac{1}{8} P_1 P_3^2 \left(\frac{\lambda}{p_0^2}\right)^{-2} \left(\frac{\lambda t}{p_0}\right)^4 \sin p_0 t - P_1 P_3 \left(\frac{\lambda}{p_0^2}\right) (1 - \cos p_0 t) \\
& + \frac{5}{2} P_1^3 \left(\frac{\lambda}{p_0^2}\right)^2 \sin p_0 t + 3P_1 \left(\frac{\lambda}{p_0^2}\right)^2 \sin p_0 t - P_1^3 \left(\frac{\lambda}{p_0^2}\right) \left(\frac{\lambda t}{p_0}\right) \\
& + \frac{1}{2} P_1 (1 - 7P_3^2) \left(\frac{\lambda}{p_0^2}\right) \left(\frac{\lambda t}{p_0}\right) \cos p_0 t
\end{aligned} \tag{47}$$

$$\begin{aligned}
e_{31} \approx\; & -P_3 \cos p_0 t - P_1^2 \left(\frac{\lambda t}{p_0}\right) \cos p_0 t + 2P_1^2 P_3 \left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t \\
& + \frac{1}{2} P_3^2 \left(\frac{\lambda}{p_0^2}\right)^{-1} \left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t + \frac{2}{3} P_1^2 P_3 \left(\frac{\lambda}{p_0^2}\right)^{-1} \left(\frac{\lambda t}{p_0}\right)^3 \sin p_0 t \\
& + \frac{1}{8} P_3^3 \left(\frac{\lambda}{p_0^2}\right)^{-2} \left(\frac{\lambda t}{p_0}\right)^4 \cos p_0 t + P_1^2 P_3 \left(\frac{\lambda}{p_0^2}\right)^2 (2 + P_1^2 + P_1 P_3 - 3P_1 P_3^3)(1 - \cos p_0 t) \\
& - \frac{7}{2} P_1^2 P_3 \left(\frac{\lambda}{p_0^2}\right) \left(\frac{\lambda t}{p_0}\right) \sin p_0 t + P_1^2 \left(\frac{\lambda}{p_0^2}\right) \sin p_0 t
\end{aligned} \tag{48}$$

$$\begin{aligned}
e_{32} \approx\; & \sin p_0 t + \frac{1}{2} P_3 \left(\frac{\lambda}{p_0^2}\right)^{-1} \left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t + \frac{1}{6} P_1^2 \left(\frac{\lambda}{p_0^2}\right)^{-1} \left(\frac{\lambda t}{p_0}\right)^3 \cos p_0 t \\
& - \frac{1}{8} P_3^2 \left(\frac{\lambda}{p_0}\right)^{-2} \left(\frac{\lambda t}{p_0}\right)^4 \sin p_0 t + \frac{1}{2} P_1^2 \left(\frac{\lambda}{p_0^2}\right) \left(\frac{\lambda t}{p_0}\right) \cos p_0 t \\
& + P_1^2 \left(\frac{\lambda}{p_0^2}\right)^2 \left(-\frac{3}{2} + P_1^2 + P_1 P_3 - 3P_1 P_3^3 \right) \sin p_0 t
\end{aligned} \tag{49}$$

$$\begin{aligned}
e_{33} \approx\; & P_1 \cos p_0 t - P_1 P_3 \left(\frac{\lambda t}{p_0}\right) \cos p_0 t - \frac{1}{2} P_1 (1 - 4P_3^2) \left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t \\
& - \frac{1}{2} P_1 P_3 \left(\frac{\lambda}{p_0^2}\right)^{-1} \left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t - \frac{1}{6} P_1 (1 - 4P_3^2) \left(\frac{\lambda}{p_0^2}\right)^{-1} \left(\frac{\lambda t}{p_0}\right)^3 \sin p_0 t \\
& - \frac{1}{8} P_1 P_3^2 \left(\frac{\lambda}{p_0^2}\right)^{-2} \left(\frac{\lambda t}{p_0}\right)^4 \cos p_0 t + P_1 P_3 \left(\frac{\lambda}{p_0^2}\right) \sin p_0 t \\
& + \frac{1}{2} P_1 (1 - 7P_3^2) \left(\frac{\lambda}{p_0^2}\right) \left(\frac{\lambda t}{p_0}\right) \sin p_0 t \\
& + P_1 \left(\frac{\lambda}{p_0^2}\right)^2 (P_1^2 P_3^2 - P_1^3 P_3 + 3P_1^3 P_3^3 + 3P_3^2)(1 - \cos p_0 t).
\end{aligned} \tag{50}$$

It is worth noting that, if $p_{30} = 0$, so that $P_1 = 1$ and $P_3 = 0$ [see equations (38)], then e_{ij} is equal to $a_{ij} (i, j = 1, 2, 3)$ because $\mathbf{e}_i$ is then equal to $\mathbf{a}_i (i = 1, 2, 3)$.

Equations (42)–(50) have been written in such a way that t is always multiplied by λ and divided by p_0 when it appears outside of the argument of a trigonometric function and λ is divided by p_0^2 when it is not multiplied by t. This representation of $e_{ij}(i, j = 1, 2, 3)$ permits one to carry out a systematic simplification of the equations and thus to arrive at approximate results which, as will be seen in the sequel, can furnish an entirely satisfactory description of the motion under consideration. Specifically, dropping from equations (42)–(50) all terms involving $(\lambda/p_0^2)^n$ with $n \geq 1$ and all terms involving $(\lambda t/p_0)^m$ with $m \geq 3$, replacing $\lambda t/p_0$ and $(\lambda t/p_0)^2$ in equations (59)–(67) with $\sin(\lambda t/p_0)$ and $2[1-\cos(\lambda t/p_0)]$, respectively, and introducing new dimensionless parameters x and z as

$$x \triangleq p_0 t, \qquad z \triangleq \frac{\lambda}{p_0^2} \tag{51}$$

one arrives at

$$e_{11} \approx k_1 - P_1 P_3 (1-\cos zx)(1-\cos x) \tag{52}$$

$$e_{12} \approx -P_1 P_3 (1-\cos zx)\sin x \tag{53}$$

$$e_{13} \approx k_3 + P_1^2 P_3 (1-\cos zx)(1-\cos x) \tag{54}$$

$$e_{21} \approx k_3 \sin x + k_4 \cos x = \sqrt{(k_3^2+k_4^2)} \cos\left[x + \tan^{-1}\left(-\frac{k_3}{k_4}\right)\right] \tag{55}$$

$$e_{22} \approx -k_5 \sin x + k_6 \cos x = \sqrt{(k_5^2+k_6^2} \cos\left[x + \tan^{-1}\left(\frac{k_5}{k_6}\right)\right] \tag{56}$$

$$e_{23} \approx -k_1 \sin x - k_2 \cos x = \sqrt{(k_1^2+k_2^2)} \cos\left[x + \tan^{-1}\left(-\frac{k_1}{k_2}\right)\right] \tag{57}$$

$$e_{31} \approx -k_3 \cos x + k_4 \sin x = \sqrt{(k_3^2+k_4^2)} \sin\left[x + \tan^{-1}\left(-\frac{k_3}{k_4}\right)\right] \tag{58}$$

$$e_{32} \approx k_5 \cos x + k_6 \sin x = \sqrt{(k_5^2+k_6^2)} \sin\left[x + \tan^{-1}\left(\frac{k_5}{k_6}\right)\right] \tag{59}$$

$$e_{33} \approx k_1 \cos x - k_2 \sin x = \sqrt{(k_1^2+k_2^2)} \sin\left[x + \tan^{-1}\left(-\frac{k_1}{k_2}\right)\right] \tag{60}$$

where $k_1, \ldots, k_6$ are defined as

$$k_1 \triangleq P_1 - P_1 P_3 \sin zx - P_1(1-4P_3^2)(1-\cos zx) \tag{61}$$

$$k_2 \triangleq P_1 P_3 z^{-1}(1-\cos zx) \tag{62}$$

$$k_3 \triangleq P_3 + P_1^2 \sin zx - 4P_1^2 P_3 (1-\cos zx) \tag{63}$$

$$k_4 \triangleq P_3^2 z^{-1}(1-\cos zx) \tag{64}$$

$$k_5 \triangleq P_3 z^{-1}(1-\cos zx) \tag{65}$$

$$k_6 \triangleq 1. \tag{66}$$

Now that approximate expressions for the direction cosines e_{ij} $(i, j = 1, 2, 3)$ are at hand, approximate expressions for angles describing the orientation of the unit vectors

$\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ relative to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ can be constructed. One set of such angles can be generated by aligning $\mathbf{c}_i$ with $\mathbf{e}_i$ ($i = 1, 2, 3$) and then performing successive right-handed rotations of C of amount θ_1 about an axis parallel to $\mathbf{c}_3$, θ_2 about an axis parallel to $\mathbf{c}_1$, and θ_3 about an axis parallel to $\mathbf{c}_3$. These angles are then related to e_{ij} ($i, j = 1, 2, 3$) as follows:

$$\sin\theta_1 = \frac{e_{13}}{\sqrt{(e_{13}^2+e_{23}^2)}}, \qquad \cos\theta_1 = \frac{-e_{23}}{\sqrt{(e_{13}^2+e_{23}^2)}} \tag{67}$$

$$\sin\theta_2 = \frac{-1}{e_{22}\sqrt{(e_{13}^2+e_{23}^2)}}(e_{13}e_{31}+e_{23}e_{32}e_{33}), \qquad \cos\theta_2 = e_{33} \tag{68}$$

$$\sin\theta_3 = \frac{e_{31}}{\sqrt{(e_{31}^2+e_{32}^2)}}, \qquad \cos\theta_3 = \frac{e_{32}}{\sqrt{(e_{31}^2+e_{32}^2)}}. \tag{69}$$

To determine the motion of the body-fixed unit vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ relative to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, it is only necessary to observe that equations (18), (38) and (51) permit one to express θ, the angle between $\mathbf{b}_1$ and $\mathbf{c}_1$, as

$$\theta = L(\tfrac{1}{2}zx^2+P_3x) \tag{70}$$

and that three angles, say φ_1, φ_2 and φ_3, analogous to θ_1, θ_2 and θ_3, but governing the orientation of $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ relative to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, are given by

$$\varphi_1 = \theta_1, \qquad \varphi_2 = \theta_2, \qquad \varphi_3 = \theta_3+\theta. \tag{71}$$

The parameters P_1 and P_3, introduced in equations (38) for purposes of analytical convenience, can be expressed in a physically more meaningful form after introducing Ω_1 and Ω_3 as

$$\Omega_1 \triangleq \frac{\omega_{10}}{\sqrt{(\omega_{10}^2+\omega_{30}^2)}}, \qquad \Omega_3 \triangleq \frac{\omega_{30}}{\sqrt{(\omega_{10}^2+\omega_{30}^2)}}. \tag{72}$$

Referring to equations (16), (23) and (72), one then obtains

$$P_1 = \frac{\Omega_1}{\sqrt{[\Omega_1^2+\Omega_3^2/(1+L)^2]}}, \qquad P_3 = \frac{\Omega_3/(1+L)}{\sqrt{[\Omega_1^2+\Omega_3^2/(1+L)^2]}}. \tag{73}$$

Evidently, if either $\Omega_1 = 1$ (so that $\Omega_3 = 0$) or $\Omega_1 = 0$ (so that $\Omega_3 = 1$), then either $P_1 = \Omega_1$ and $P_3 = 0$ or $P_1 = 0$ and $P_3 = \Omega_3$, regardless of the value of L. This means that the initial angular velocity of C is equal to that of B if B initially has a pure tumbling or pure spinning motion.

In summary, given the dimensionless parameters Ω_1, Ω_3, L, z and x, one proceeds as follows to evaluate φ_1, φ_2 and φ_3:

1. compute P_1 and P_3 from equations (73);
2. use equations (61)–(66) to evaluate $k_1, \ldots, k_6$;
3. find e_{ij} ($i, j = 1, 2, 3$) from equations (52)–(60);
4. evaluate θ_1, θ_2 and θ_3 by reference to equations (67)–(69);
5. find θ from equation (70);
6. determine φ_1, φ_2 and φ_3 by using equations (71).

A measure of the utility of the procedure described above may be obtained by comparing values of θ_1, θ_2 and θ_3 found by means of this procedure with values resulting from a numerical integration of equations (33)–(35), subsequent evaluation of e_{ij} $(i, j = 1, 2, 3)$ by reference to equation (41), and determination of θ_1, θ_2 and θ_3 by inversion of equations (67)–(69). Figures 2–5 each show θ_1, θ_2 and θ_3 as functions of x, the values corresponding to the digital computer solution of the exact equations being represented by solid curves, whereas the approximate solution is represented by crosses; and each figure applies to a different combination of values of Ω_1, Ω_3, L and z.

The accuracy of the approximate solutions is seen to be relatively insensitive to the numerical values of Ω_1, Ω_3 (see Figs. 2–4) and L (see Figs. 3 and 4), but to decrease both

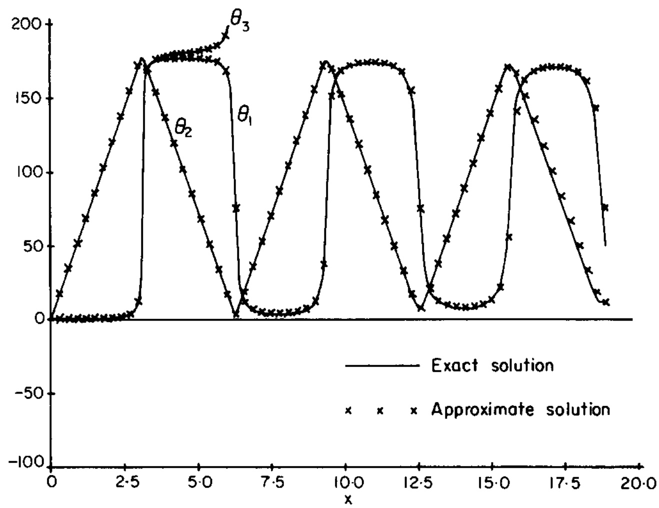

FIG. 2. Comparison of solutions for orientation angles. $\Omega_1 = 1{\cdot}0$, $\Omega_3 = 0$, $z = 0{\cdot}01$.

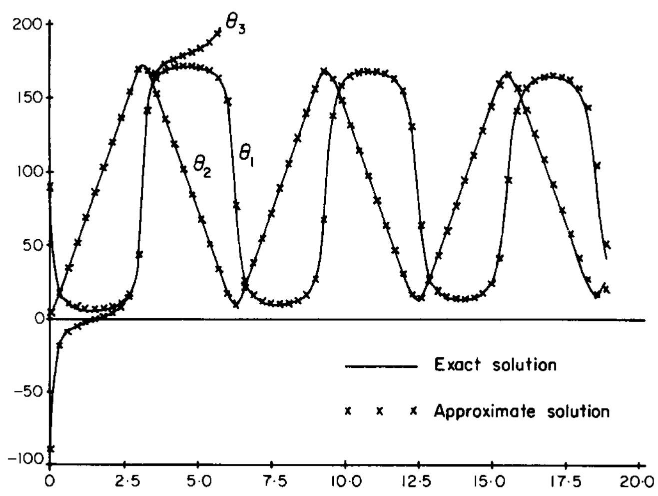

FIG. 3. Comparison of solutions for orientation angles. $\Omega_1 = 0{\cdot}70711$, $\Omega_3 = 0{\cdot}70711$, $z = 0{\cdot}01$, $L = 9{\cdot}0$.

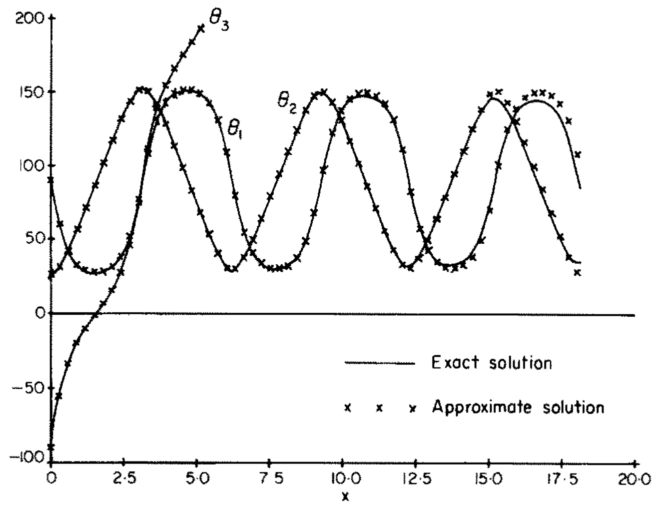

FIG. 4. Comparison of solutions for orientation angles. $\Omega_1 = 0{\cdot}70711$, $\Omega_3 = 0{\cdot}070711$, $z = 0{\cdot}01$, $L = 1{\cdot}0$.

when z increases (cf. Fig. 3 with Fig. 5) and when x increases (see Figs. 2–5). We note for future reference that for $z = 0{\cdot}01$ (see Figs. 2–4) the exact and the approximate value of θ_i $(i = 1, 2, 3)$ are nearly indistinguishable from each other so long as x remains smaller than 15·00 and that similarly good agreement obtains for $z = 0{\cdot}05$ (see Fig. 5) for $x < 2{\cdot}5$.

TRANSLATIONAL MOTION

To study the translational motion of B, that is, the motion of the mass center B^* of B, we recall the definitions of m, $\mathbf{X}$ and $\mathbf{F}$; and, after defining X_i $(i = 1, 2, 3)$ and f_i $(i = 1, 2)$ as

$$X_i \triangleq \mathbf{X} \, . \, \mathbf{e}_i \tag{74}$$

$$f_i \triangleq F_i/m \tag{75}$$

express Newton's Second Law of Motion in the form

$$\ddot{X}_1 = [e_{11}(f_1 \cos\theta - f_2 \sin\theta) + e_{12}(f_1 \sin\theta + f_2 \cos\theta)] \tag{76}$$

$$\ddot{X}_2 = [e_{21}(f_1 \cos\theta - f_2 \sin\theta) + e_{22}(f_1 \sin\theta + f_2 \cos\theta)] \tag{77}$$

$$\ddot{X}_3 = [e_{31}(f_1 \cos\theta - f_2 \sin\theta) + e_{32}(f_1 \sin\theta + f_2 \cos\theta)]. \tag{78}$$

Equations (76)–(78) are coupled to the rotational motion of B by the quantities θ and e_{ij} $(i, j = 1, 2, 3)$. If these are eliminated by using equations (70) and (52)–(60) and if new dimensionless quantities D_i, $\mathscr{F}_i$, ζ and β_i are defined as

$$D_i \triangleq X_i \left(\frac{2J}{m}\right)^{-\frac{1}{2}} \qquad (i = 1, 2, 3) \tag{79}$$

$$\mathscr{F}_i \triangleq \frac{f_i}{p_0^2 \sqrt{(2J/m)}} \qquad (i = 1, 2, 3) \tag{80}$$

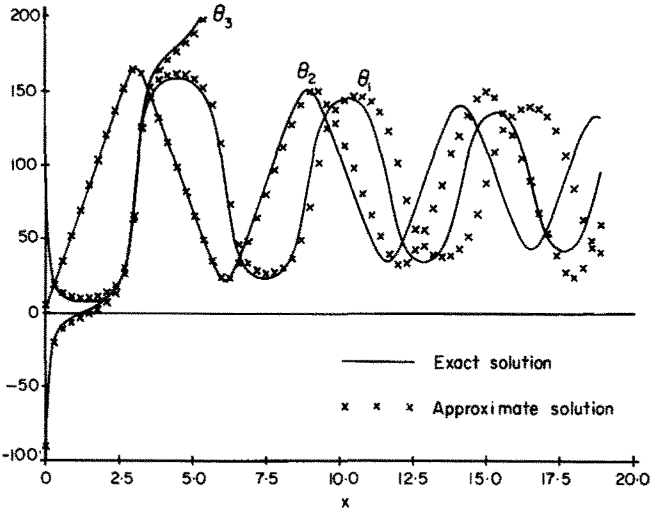

FIG. 5. Comparison of solutions for orientation angles. $\Omega_1 = 0{\cdot}70711$, $\Omega_3 = 0{\cdot}70711$, $z = 0{\cdot}05$, $L = 9{\cdot}0$.

$$\zeta \triangleq \tfrac{1}{2}Lz \tag{81}$$

$$\beta_1 \triangleq \tfrac{1}{2}LP_3 \tag{82}$$

$$\beta_2 \triangleq (LP_3+z)/2 \tag{83}$$

$$\beta_3 \triangleq (LP_3-z)/2 \tag{84}$$

$$\beta_4 \triangleq (LP_3-1)/2 \tag{85}$$

$$\beta_5 \triangleq (LP_3+1)/2 \tag{86}$$

$$\beta_6 \triangleq (LP_3-z-1)/2 \tag{87}$$

$$\beta_7 \triangleq (LP_3-z+1)/2 \tag{88}$$

$$\beta_8 \triangleq (LP_3+z-1)/2 \tag{89}$$

$$\beta_9 \triangleq (LP_3+z+1)/3 \tag{90}$$

then, when primes are used to denote differentiation with respect to x, D_i ($i = 1,2,3$) are found to be governed by

$$D_1'' \approx \mathscr{F}_1 P_1 \Big\{(4P_3^2-P_3)\cos(\zeta x^2+2\beta_1 x) - \frac{P_3}{2}[\sin(\zeta x^2+2\beta_2 x)-\sin(\zeta x^2+2\beta_3 x)$$
$$+\cos(\zeta x^2+2\beta_7 x)+\cos(\zeta x^2+2\beta_9 x)]+P_3\cos(\zeta x^2+2\beta_5 x)$$
$$+\tfrac{1}{2}(1-4P_3^2+P_3)[\cos(\zeta x^2+2\beta_3 x)+\cos(\zeta x^2+2\beta_2 x)]\Big\}$$
$$-\mathscr{F}_2 P_1 \Big\{(4P_3^2-P_3)\sin(\zeta x^2+2\beta_1 x) - \frac{P_3}{2}[\cos(\zeta x^2+2\beta_3 x)-\cos(\zeta x^2+2\beta_2 x)$$

$$+\sin(\zeta x^2+2\beta_7 x)+\sin(\zeta x^2+2\beta_9 x)]+P_3\sin(\zeta x^2+2\beta_5 x)$$

$$\left.+\tfrac{1}{2}(1-4P_3^2+P_3)[\sin(\zeta x^2+2\beta_2 x)+\sin(\zeta x^2+2\beta_3 x)]\right\} \tag{91}$$

$$D_2'' \approx \mathscr{F}_1\left\{\tfrac{1}{2}(P_3-4P_1^2P_3)[\sin(\zeta x^2+2\beta_5 x)-\sin(\zeta x^2+2\beta_4 x)]+\frac{P_1^2}{4}[\cos(\zeta x^2+2\beta_8 x)\right.$$

$$-\cos(\zeta x^2+2\beta_9 x)-\cos(\zeta x^2+2\beta_6 x)+\cos(\zeta x^2+2\beta_7 x)]$$

$$+P_1^2P_3[\sin(\zeta x^2+2\beta_7 x)-\sin(\zeta x^2+2\beta_6 x)+\sin(\zeta x^2+2\beta_9 x)$$

$$\left.-\sin(\zeta x^2+2\beta_8 x)]\right\}+\mathscr{F}_1P_3^2z^{-1}\left\{\tfrac{1}{2}[\cos(\zeta x^2+2\beta_4 x)+\cos(\zeta x^2+2\beta_5 x)]\right.$$

$$\left.-\tfrac{1}{4}[\cos(\zeta x^2+2\beta_6 x)+\cos(\zeta x^2+2\beta_7 x)+\cos(\zeta x^2+2\beta_8 x)+\cos(\zeta x^2+2\beta_9 x)]\right\}$$

$$-\mathscr{F}_2\left\{\tfrac{1}{2}(P_3-4P_1^2P_3)[\cos(\zeta x^2+2\beta_4 x)-\cos(\zeta x^2+2\beta_5 x)]\right.$$

$$+\frac{P_1^2}{4}[\sin(\zeta x^2+2\beta_7 x)-\sin(\zeta x^2+2\beta_6 x)-\sin(\zeta x^2+2\beta_9 x)$$

$$+\sin(\zeta x^2+2\beta_8 x)+P_1^2P_3[\cos(\zeta x^2+2\beta_8 x)-\cos(\zeta x^2+2\beta_9 x)$$

$$\left.+\cos(\zeta x^2+2\beta_6 x)-\cos(\zeta x^2+2\beta_7 x)]\right\}-\mathscr{F}_2P_3^2z^{-1}\left\{\tfrac{1}{2}[\sin(\zeta x^2+2\beta_4 x)\right.$$

$$+\sin(\zeta x^2+2\beta_5 x)]-\tfrac{1}{4}[\sin(\zeta x^2+2\beta_9 x)+\sin(\zeta x^2+2\beta_8 x)$$

$$\left.+\sin(\zeta x^2+2\beta_7 x)+\sin(\zeta x^2+2\beta_6 x)]\right\}+\mathscr{F}_1\left\{\tfrac{1}{2}[\sin(\zeta x^2+2\beta_4 x)\right.$$

$$+\sin(\zeta x^2+2\beta_5 x)]-\frac{P_3}{2}z^{-1}[\cos(\zeta x^2+2\beta_4 x)-\cos(\zeta x^2+2\beta_5 x)]$$

$$+\frac{P_3}{4}z^{-1}[\cos(\zeta x^2+2\beta_8 x)-\cos(\zeta x^2+2\beta_9 x)+\cos(\zeta x^2+2\beta_6 x)$$

$$\left.-\cos(\zeta x^2+2\beta_7 x)]\right\}+\mathscr{F}_2\left\{\tfrac{1}{2}[\cos(\zeta x^2+2\beta_4 x)+\cos(\zeta x^2+2\beta_5 x)]\right.$$

$$-\frac{P_3}{2}z^{-1}[\sin(\zeta x^2+2\beta_5 x)-\sin(\zeta x^2+2\beta_4 x)]+\frac{P_3}{4}z^{-1}[\sin(\zeta x^2+2\beta_7 x)$$

$$\left.-\sin(\zeta x^2+2\beta_6 x)+\sin(\zeta x^2+2\beta_9 x)-\sin(\zeta x^2+2\beta_8 x)]\right\} \qquad z\neq 0 \tag{92}$$

$$D_3'' \approx \mathscr{F}_1\left\{\frac{P_3}{2}(-1+4P_1^2)[\cos(\zeta x^2+2\beta_4 x)+\cos(\zeta x^2+2\beta_5 x)]\right.$$

$$-\frac{P_1^2}{4}[\sin(\zeta x^2+2\beta_9 x)+\sin(\zeta x^2+2\beta_8 x)-\sin(\zeta x^2+2\beta_7 x)$$

$$-\sin(\zeta x^2+2\beta_6 x)-P_1^2P_3[\cos(\zeta x^2+2\beta_6 x)+\cos(\zeta x^2+2\beta_7 x)$$

$$
\begin{aligned}
&\left.+\cos(\zeta x^2+2\beta_8 x)+\cos(\zeta x^2+2\beta_9 x)]\right\} + \mathscr{F}_1 P_3^2 z^{-1}\left\{\tfrac{1}{2}[\sin(\zeta x^2+2\beta_5 x)\right. \\
&-\sin(\zeta x^2+2\beta_4 x)]-\tfrac{1}{4}[\sin(\zeta x^2+2\beta_7 x)-\sin(\zeta x^2+2\beta_6 x)+\sin(\zeta x^2+2\beta_9 x) \\
&\left.-\sin(\zeta x^2+2\beta_8 x)]\right\}-\mathscr{F}_2\left\{\frac{P_3}{2}(-1+4P_1^2)[\sin(\zeta x^2+2\beta_4 x)+\sin(\zeta x^2+2\beta_5 x)]\right. \\
&-\frac{P_1^2}{4}[\cos(\zeta x^2+2\beta_6 x)+\cos(\zeta x^2+2\beta_7 x)-\cos(\zeta x^2+2\beta_8 x) \\
&-\cos(\zeta x^2+2\beta_9 x)]-P_1^2 P_3[\sin(\zeta x^2+2\beta_9 x)+\sin(\zeta x^2+2\beta_8 x) \\
&\left.+\sin(\zeta x^2+2\beta_7 x)+\sin(\zeta x^2+2\beta_6 x)]\right\}-\mathscr{F}_2 P_3^2 z^{-1}\{\tfrac{1}{2}[\cos(\zeta x^2+2\beta_4 x) \\
&-\cos(\zeta x^2+2\beta_5 x)]-\tfrac{1}{4}[\cos(\zeta x^2+2\beta_8 x)-\cos(\zeta x^2+2\beta_9 x) \\
&\left.+\cos(\zeta x^2+2\beta_6 x)-\cos(\zeta x^2+2\beta_7 x)]\right\}+\mathscr{F}_1\left\{\frac{P_3}{2}z^{-1}[\sin(\zeta x^2+2\beta_4 x)\right. \\
&+\sin(\zeta x^2+2\beta_5 x)]+\tfrac{1}{2}[\cos(\zeta x^2+2\beta_4 x)-\cos(\zeta x^2+2\beta_5 x)] \\
&-\frac{P_3}{4}z^{-1}[\sin(\zeta x^2+2\beta_9 x)+\sin(\zeta x^2+2\beta_8 x)+\sin(\zeta x^2+2\beta_7 x) \\
&\left.+\sin(\zeta x^2+2\beta_6 x)]\right\}+\mathscr{F}_2\left\{\frac{P_3}{2}z^{-1}[\cos(\zeta x^2+2\beta_4 x)+\cos(\zeta x^2+2\beta_5 x)]\right. \\
&+\tfrac{1}{2}[\sin(\zeta x^2+2\beta_5 x)-\sin(\zeta x^2+2\beta_4 x)]-\frac{P_3}{4}z^{-1}[\cos(\zeta x^2+2\beta_6 x) \\
&\left.+\cos(\zeta x^2+2\beta_7 x)+\cos(\zeta x^2+2\beta_8 x)+\cos(\zeta x^2+2\beta_9 x)]\right\} \qquad z \neq 0. \qquad (93)
\end{aligned}
$$

Furthermore, $D_i(x)$ $(i = 1, 2, 3)$ must satisfy the initial conditions

$$D_i(0) = 0, \qquad D_i'(0) = V_{i0} \qquad (i = 1, 2, 3). \tag{94}$$

[The reason for taking $D_i(0)$ $(i = 1, 2, 3)$ equal to zero is that no loss of generality results from regarding B^* as initially coincident with point 0 (see Fig. 1), since the choice of point 0 is arbitrary.]

The solution of equations (91)–(93) can be expressed in terms of certain functions related to Fresnel integrals. The functions to be employed are defined as follows:

$$C_2[w] \triangleq \frac{1}{\sqrt{(2\pi)}}\int_0^w \frac{\cos u}{\sqrt{u}}\,du \tag{95}$$

$$S_2[w] \triangleq \frac{1}{\sqrt{(2\pi)}}\int_0^w \frac{\sin u}{\sqrt{u}}\,du \tag{96}$$

$$C(a,b,x) \triangleq \left[\frac{\pi}{2a}\sqrt{\left(\frac{2}{a\pi}\right)}(ax+b)\right]\left\{\cos\left(\frac{b^2}{a}\right)C_2\left[\frac{1}{a}(ax+b)^2\right]\right.$$
$$\left.+\sin\left(\frac{b^2}{a}\right)S_2\left[\frac{1}{a}(ax+b)^2\right]\right\}-\frac{1}{2a}\sin(ax^2+2bx) \qquad a>0 \tag{97}$$

$$S(a,b,x) \triangleq \left[\frac{\pi}{2a}\sqrt{\left(\frac{2}{a\pi}\right)}(ax+b)\right]\left\{\cos\left(\frac{b^2}{a}\right)S_2\left[\frac{1}{a}(ax+b)^2\right]\right.$$
$$\left.-\sin\left(\frac{b^2}{a}\right)C_2\left[\frac{1}{a}(ax+b)^2\right]\right\}+\frac{1}{2a}\cos(ax^2+2bx) \qquad a>0 \tag{98}$$

$$C_1(a,b,x) \triangleq \sqrt{\left(\frac{\pi}{2a}\right)}\left\{\cos\left(\frac{b^2}{a}\right)C_2\left[\frac{1}{a}(ax+b)^2\right]+\sin\left(\frac{b^2}{a}\right)S_2\left[\frac{1}{a}(ax+b)^2\right]\right\} \qquad a>0 \tag{99}$$

$$S_1(a,b,x) \triangleq \sqrt{\left(\frac{\pi}{2a}\right)}\left\{\cos\left(\frac{b^2}{a}\right)S_2\left[\frac{1}{a}(ax+b)^2\right]-\sin\left(\frac{b^2}{a}\right)C_2\left[\frac{1}{a}(ax+b)^2\right]\right\} \qquad a>0 \tag{100}$$

$$C^*(a,b,x) \triangleq \int_0^x\int_0^v \cos(au^2+2bu)\,du\,dv$$
$$= C(a,b,x)-C_1(a,b,0)x-C(a,b,0) \qquad a>0 \tag{101}$$

$$S^*(a,b,x) \triangleq \int_0^x\int_0^v \sin(au^2+2bu)\,du\,dv$$
$$= S(a,b,x)-S_1(a,b,0)x-S(a,b,0) \qquad a>0. \tag{102}$$

Two successive integrations of equations (91)–(93) and use of equations (94) now lead to the following expression for D_1, D_2 and D_3:

$$D_1 = \mathscr{F}_1 P_1\left\{(4P_3^2-P_3)C^*(\zeta,\beta_1,x)-\frac{P_3}{2}[S^*(\zeta,\beta_2,x)-S^*(\zeta,\beta_3,x)+C^*(\zeta,\beta_7,x)\right.$$
$$\left.+C^*(\zeta,\beta_9,x)]+P_3C^*(\zeta,\beta_5,x)-\tfrac{1}{2}(1-4P_3^2+P_3)[C^*(\zeta,\beta_3,x)+C^*(\zeta,\beta_2,x]\right\}$$
$$-\mathscr{F}_2 P_1\left\{(4P_3^2-P_3)S^*(\zeta,\beta_1,x)-\frac{P_3}{2}[C^*(\zeta,\beta_3,x)-C^*(\zeta,\beta_2,x)+S^*(\zeta,\beta_7,x)\right.$$
$$\left.+S^*(\zeta,\beta_9,x)+P_3S^*(\zeta,\beta_5,x)+\tfrac{1}{2}(1-4P_3^2+P_3)[S^*(\zeta,\beta_2,x)+S^*(\zeta,\beta_3,x)]\right\}$$
$$+V_{10}x \qquad \zeta>0 \tag{103}$$

$$D_2 \approx \mathscr{F}_1\left\{\tfrac{1}{2}(P_3-4P_1^2P_3)[S^*(\zeta,\beta_5,x)-S^*(\zeta,\beta_4,x)]+\frac{P_1^2}{4}[C^*(\zeta,\beta_8,x)-C^*(\zeta,\beta_9,x)\right.$$
$$-C^*(\zeta,\beta_6,x)+C^*(\zeta,\beta_7,x)]+P_1^2P_3[S^*(\zeta,\beta_7,x)-S^*(\zeta,\beta_6,x)+S^*(\zeta,\beta_9,x)$$
$$\left.-S^*(\zeta,\beta_8,x)]\right\}+\mathscr{F}_1P_3^2z^{-1}\left\{\tfrac{1}{2}[C^*(\zeta,\beta_4,x)+C^*(\zeta,\beta_5,x)]-\tfrac{1}{4}[C^*(\zeta,\beta_6,x)\right.$$
$$\left.+C^*(\zeta,\beta_7,x)-C^*(\zeta,\beta_8,x)+C^*(\zeta,\beta_9,x)]\right\}-\mathscr{F}_2\left\{\tfrac{1}{2}(P_3-4P_1^2P_3)[C^*(\zeta,\beta_4,x)\right.$$

$$
\begin{aligned}
&-C^*(\zeta,\beta_5,x)]+\frac{P_1^2}{4}[S^*(\zeta,\beta_7,x)-S^*(\zeta,\beta_6,x)-S^*(\zeta,\beta_9,x)+S^*(\zeta,\beta_8,x)]\\
&+P_1^2P_3[C^*(\zeta,\beta_8,x)-C^*(\zeta,\beta_9,x)+C^*(\zeta,\beta_6,x)-C^*(\zeta,\beta_7,x)]\Big\}-\mathscr{F}_2P_3^2z^{-1}\\
&\times\Big\{\tfrac{1}{2}[S^*(\zeta,\beta_4,x)+S^*(\zeta,\beta_5,x)]-\tfrac{1}{4}[S^*(\zeta,\beta_9,x)+S^*(\zeta,\beta_8,x)+S^*(\zeta,\beta_7,x)\\
&+S^*(\zeta,\beta_6,x)]\Big\}+\mathscr{F}_1\Big\{\tfrac{1}{2}[S^*(\zeta,\beta_4,x)+S^*(\zeta,\beta_5,x)]-\frac{P_3}{2}z^{-1}[C^*(\zeta,\beta_4,x)\\
&-C^*(\zeta,\beta_5,x)]+\frac{P_3}{4}z^{-1}[C^*(\zeta,\beta_8,x)-C^*(\zeta,\beta_9,x)+C^*(\zeta,\beta_6,x)-C^*(\zeta,\beta_7,x)]\Big\}\\
&+\mathscr{F}_2\Big\{\tfrac{1}{2}[C^*(\zeta,\beta_4,x)+C^*(\zeta,\beta_5,x)]-\frac{P_3}{2}z^{-1}[S^*(\zeta,\beta_5,x)-S^*(\zeta,\beta_4,x)]\\
&+\frac{P_3}{4}z^{-1}[S^*(\zeta,\beta_7,x)-S^*(\zeta,\beta_6,x)+S^*(\zeta,\beta_9,x)-S^*(\zeta,\beta_8,x)]\Big\}+V_{20}x\qquad\zeta>0 \qquad (104)
\end{aligned}
$$

$$
\begin{aligned}
D_3\approx\;&\mathscr{F}_1\Big\{\frac{P_3}{2}(-1+4P_1^2)[C^*(\zeta,\beta_4,x)+C^*(\zeta,\beta_5,x)]-\frac{P_1^2}{4}[S^*(\zeta,\beta_9,x)+S^*(\zeta,\beta_8,x)\\
&-S^*(\zeta,\beta_7,x)-S^*(\zeta,\beta_6,x)]-P_1^2P_3[C^*(\zeta,\beta_6,x)+C^*(\zeta,\beta_7,x)+C^*(\zeta,\beta_8,x)+C^*(\zeta,\beta_9,x)]\Big\}\\
&+\mathscr{F}_1P_3^2z^{-1}\Big\{\tfrac{1}{2}[S^*(\zeta,\beta_5,x)-S^*(\zeta,\beta_4,x)]-\tfrac{1}{4}S^*(\zeta,\beta_7,x)-S^*(\zeta,\beta_6,x)+S^*(\zeta,\beta_9,x)\\
&-S^*(\zeta,\beta_8,x)]\Big\}-\mathscr{F}_2\Big\{\frac{P_3}{2}(-1+4P_1^2)[S^*(\zeta,\beta_4,x)+S^*(\zeta,\beta_5,x)]-\frac{P_1^2}{4}[C^*(\zeta,\beta_6,x)\\
&+C^*(\zeta,\beta_7,x)-C^*(\zeta,\beta_8,x)-C^*(\zeta,\beta_9,x)]-P_1^2P_3[S^*(\zeta,\beta_9,x)+S^*(\zeta,\beta_8,x)\\
&+S^*(\zeta,\beta_7,x)+S^*(\zeta,\beta_6,x)]\Big\}-\mathscr{F}_2P_3^2z^{-1}\Big\{\tfrac{1}{2}[C^*(\zeta,\beta_4,x)-C^*(\zeta,\beta_5,x)]\\
&-\tfrac{1}{4}[C^*(\zeta,\beta_8,x)-C^*(\zeta,\beta_9,x)+C^*(\zeta,\beta_6,x)-C^*(\zeta,\beta_7,x)]\Big\}\\
&+\mathscr{F}_1\Big\{\frac{P_3}{2}z^{-1}[S^*(\zeta,\beta_4,x)+S^*(\zeta,\beta_5,x)]+\tfrac{1}{2}[C^*(\zeta,\beta_4,x)-C^*(\zeta,\beta_5,x)]\\
&-\frac{P_3}{4}z^{-1}[S^*(\zeta,\beta_9,x)+S^*(\zeta,\beta_8,x)+S^*(\zeta,\beta_7,x)+S^*(\zeta,\beta_6,x)]\Big\}\\
&+\mathscr{F}_2\Big\{\frac{P_3}{2}z^{-1}[C^*(\zeta,\beta_4,x)+C^*(\zeta,\beta_5,x)]+\tfrac{1}{2}[S^*(\zeta,\beta_5,x)-S^*(\zeta,\beta_4,x)]\\
&-\frac{P_3}{4}z^{-1}[C^*(\zeta,\beta_6,x)+C^*(\zeta,\beta_7,x)+C^*(\zeta,\beta_8,x)+C^*(\zeta,\beta_9,x)]\Big\}+V_{30}x\quad\zeta>1. \qquad (105)
\end{aligned}
$$

Equations (103)–(105) are valid only for $\zeta > 0$. To deal with $\zeta < 0$, parameters $\bar{\zeta}$ and $\bar{\beta}_i (i = 1, \ldots, 9)$ are introduced as

$$\bar{\zeta} \triangleq -\zeta, \qquad \bar{\beta}_i \triangleq -\beta_i \qquad (i = 1, \ldots, 9) \qquad (106)$$

which permits one to express the solution to equations (91)–(93) satisfying equations (94) as follows:

$$D_1 \approx \mathscr{F}_1 P_1 \Big\{(4P_3^2 - P_3)C^*(\bar{\zeta}, \bar{\beta}_1, x) - \frac{P_3}{2}[-S^*(\bar{\zeta}, \bar{\beta}_2, x) + S^*(\bar{\zeta}, \bar{\beta}_3, x) + C^*(\bar{\zeta}, \bar{\beta}_7, x)$$
$$+ C^*(\bar{\zeta}, \bar{\beta}_9, x)] + P_3 C^*(\bar{\zeta}, \bar{\beta}_5, x) + \tfrac{1}{2}(1 - 4P_3^2 + P_3)[C^*(\bar{\zeta}, \bar{\beta}_3, x) + C^*(\bar{\zeta}, \bar{\beta}_2, x)]\Big\}$$
$$- \mathscr{F}_2 P_1 \Big\{-(4P_3^2 - P_3)S^*(\bar{\zeta}, \bar{\beta}_1, x) - \frac{P_3}{2}[C^*(\bar{\zeta}, \bar{\beta}_3, x) - C^*(\bar{\zeta}, \bar{\beta}_2, x) - S^*(\bar{\zeta}, \bar{\beta}_7, x)$$
$$- S^*(\bar{\zeta}, \bar{\beta}_9, x)] - P_3 S^*(\bar{\zeta}, \bar{\beta}_5, x) - \tfrac{1}{2}(1 - 4P_3^2 + P_3)[S^*(\bar{\zeta}, \bar{\beta}_2, x) + S^*(\bar{\zeta}, \bar{\beta}_3, x)]\Big\}$$
$$+ V_{10} x \qquad \zeta < 0 \tag{107}$$

$$D_2 \approx \mathscr{F}_1 \Big\{\tfrac{1}{2}(P_3 - 4P_1^2 P_3)[-S^*(\bar{\zeta}, \bar{\beta}_5, x) + S^*(\bar{\zeta}, \bar{\beta}_4, x)] + \frac{P_1^2}{4}[C^*(\bar{\zeta}, \bar{\beta}_8, x) - C^*(\bar{\zeta}, \bar{\beta}_9, x)$$
$$- C^*(\bar{\zeta}, \bar{\beta}_6, x) + C^*(\bar{\zeta}, \bar{\beta}_7, x)] - P_1^2 P_3[S^*(\bar{\zeta}, \bar{\beta}_7, x) - S^*(\bar{\zeta}. \bar{\beta}_6, x) + S^*(\bar{\zeta}, \bar{\beta}_9, x)$$
$$- S^*(\bar{\zeta}, \bar{\beta}_8, x)]\Big\} + \mathscr{F}_1 P_3^2 z^{-1}\Big\{\tfrac{1}{2}[C^*(\bar{\zeta}, \bar{\beta}_4, x) + C^*(\bar{\zeta}, \bar{\beta}_5, x)] - \tfrac{1}{4}[C^*(\bar{\zeta}, \bar{\beta}_6, x)$$
$$+ C^*(\bar{\zeta}, \bar{\beta}_7, x) + C^*(\bar{\zeta}, \bar{\beta}_8, x) + C^*(\bar{\zeta}, \bar{\beta}_9, x)]\Big\} - \mathscr{F}_2 \Big\{\tfrac{1}{2}(P_3 - 4P_1^2 P_3)[C^*(\bar{\zeta}, \bar{\beta}_4, x)$$
$$- C^*(\bar{\zeta}, \bar{\beta}_5, x)] - \frac{P_1^2}{4}[S^*(\bar{\zeta}, \bar{\beta}_7, x) - S^*(\bar{\zeta}, \bar{\beta}_6, x) - S^*(\bar{\zeta}, \bar{\beta}_9, x) + S^*(\bar{\zeta}, \bar{\beta}_8, x)]$$
$$+ P_1^2 P_3[C^*(\bar{\zeta}, \bar{\beta}_8, x) - C^*(\bar{\zeta}, \bar{\beta}_9, x) + C^*(\bar{\zeta}, \bar{\beta}_6, x) - C^*(\bar{\zeta}, \bar{\beta}_7, x)]\Big\}$$
$$- \mathscr{F}_2 P_3^2 z^{-1}\Big\{-\tfrac{1}{2}[S^*(\bar{\zeta}, \bar{\beta}_4, x) + S^*(\bar{\zeta}, \bar{\beta}_5, x)] + \tfrac{1}{4}[S^*(\bar{\zeta}, \bar{\beta}_9, x) + S^*(\bar{\zeta}, \bar{\beta}_8, x)$$
$$+ S^*(\bar{\zeta}, \bar{\beta}_7, x) + S^*(\bar{\zeta}, \bar{\beta}_6, x)]\Big\} + \mathscr{F}_1 \Big\{-\tfrac{1}{2}[S^*(\bar{\zeta}, \bar{\beta}_4, x) + S^*(\bar{\zeta}, \bar{\beta}_5, x)] - \frac{P_3}{2} z^{-1}[C^*(\bar{\zeta}, \bar{\beta}_4, x)$$
$$- C^*(\bar{\zeta}, \bar{\beta}_5, x)] + \frac{P_3}{4} z^{-1}[C^*(\bar{\zeta}, \bar{\beta}_8, x) - C^*(\bar{\zeta}, \bar{\beta}_9, x) + C^*(\bar{\zeta}, \bar{\beta}_6, x) - C^*(\bar{\zeta}\, \bar{\beta}_7, x)]\Big\}$$
$$+ \mathscr{F}_2 \Big\{\tfrac{1}{2}[C^*(\bar{\zeta}, \bar{\beta}_4, x) + C^*(\bar{\zeta}, \bar{\beta}_5, x)] + \frac{P_3}{2} z^{-1}[S^*(\bar{\zeta}, \bar{\beta}_5, x) - S^*(\bar{\zeta}, \bar{\beta}_4, x)]$$
$$- \frac{P_3}{4} z^{-1}[S^*(\bar{\zeta}, \bar{\beta}_7, x) - S^*(\bar{\zeta}. \bar{\beta}_6, x) + S^*(\bar{\zeta}, \bar{\beta}_9, x) - S^*(\bar{\zeta}. \bar{\beta}_6, x)]\Big\}$$
$$+ V_{20} x \qquad \zeta < 0 \tag{108}$$

$$D_3 = \mathscr{F}_1 \Big\{\frac{P_3}{2}(-1 + 4P_1^2)[C^*(\bar{\zeta}, \bar{\beta}_4, x) + C^*(\bar{\zeta}, \bar{\beta}_5, x)] + \frac{P_1^2}{4}[S^*(\bar{\zeta}, \bar{\beta}_9, x) + S^*(\bar{\xi}, \bar{\beta}_8, x)]$$
$$- S^*(\bar{\zeta}, \bar{\beta}_7, x) - S^*(\bar{\zeta}, \bar{\beta}_6, x)] - P_1^2 P_3[C^*(\bar{\zeta}, \bar{\beta}_6, x) + C^*(\bar{\zeta}, \bar{\beta}_7, x) + C^*(\bar{\zeta}, \bar{\beta}_8, x)$$

$$
\begin{aligned}
&+C^*(\zeta,\bar\beta_9,x)]\Big\} + \mathscr{F}_1 P_3^2 z^{-1}\Big\{-\tfrac{1}{2}[S^*(\bar\zeta,\bar\beta_5,x)-S^*(\bar\zeta,\bar\beta_4,x)]+\tfrac{1}{4}[S^*(\zeta,\bar\beta_7,x)\\
&-S^*(\zeta,\bar\beta_6,x)+S^*(\zeta,\bar\beta_9,x)-S^*(\bar\zeta,\bar\beta_8,x)]\Big\} - \mathscr{F}_2\Big\{-\frac{P_3}{2}(-1+4P_1^2)[S^*(\bar\zeta,\bar\beta_4,x)\\
&+S^*(\zeta,\bar\beta_5,x)] - \frac{P_1^2}{4}[C^*(\zeta,\bar\beta_6,x)+C^*(\zeta_7,x)-C^*(\zeta,\bar\beta_8,x)-C^*(\zeta,\bar\beta_9,x)]\\
&+P_1^2P_3[S^*(\zeta,\bar\beta_9,x)+S^*(\zeta,\bar\beta_8,x)+S^*(\zeta,\bar\beta_7,x)+S^*(\bar\zeta,\bar\beta_6,x)]\Big\}\\
&-\mathscr{F}_2 P_3^2 z^{-1}\{\tfrac{1}{2}[C^*(\bar\zeta,\bar\beta_4,x)-C^*(\zeta,\bar\beta_5,x)]-\tfrac{1}{4}[C^*(\zeta,\bar\beta_8,x)-C^*(\zeta,\bar\beta_9,x)\\
&+C^*(\zeta,\bar\beta_6,x)-C^*(\zeta,\bar\beta_7,x)]\Big\} + \mathscr{F}_1\Big\{-\frac{P_3}{2}z^{-1}[S^*(\zeta,\bar\beta_4,x)+S^*(\zeta,\bar\beta_5,x)]\\
&+\tfrac{1}{2}[C^*(\zeta,\bar\beta_4,x)-C^*(\zeta,\bar\beta_5,x)]+\frac{P_3}{4}z^{-1}[S^*(\zeta,\bar\beta_9,x)+S^*(\bar\zeta,\bar\beta_8,x)+S^*(\bar\zeta,\bar\beta_7,x)\\
&+S^*(\bar\zeta,\bar\beta_6,x)]\Big\} + \mathscr{F}_2\Big\{\frac{P_3}{2}z^{-1}[C^*(\zeta,\bar\beta_4,x)+C^*(\zeta,\bar\beta_5,x)]-\tfrac{1}{2}[S^*(\bar\zeta,\bar\beta_5,x)\\
&-S^*(\bar\zeta,\bar\beta_4,x)]-\frac{P_3}{4}z^{-1}[C^*(\bar\zeta,\bar\beta_6,x)+C^*(\bar\zeta,\bar\beta_7,x)+C^*(\bar\zeta,\bar\beta_8,x)+C^*(\bar\zeta,\bar\beta_9,x]\Big\}\\
&+V_{30}x \qquad \zeta < 0. \qquad (109)
\end{aligned}
$$

Finally, for $\zeta = 0$, the solution to equations (91)–(93) that satisfies the initial conditions in equation (94) is

$$
\begin{aligned}
D_1 \approx{}& \mathscr{F}_1 P_1\Big\{(4P_3^2-P_3)\left(\frac{x^2}{2}\right)-\frac{P_3}{2}[S^*(\beta_2,x)-S^*(\beta_3,x)+C^*(\beta_7,x)+C^*(\beta_9,x)]\\
&+P_3C^*(\beta_5,x)+\tfrac{1}{2}(1-4P_3^2+P_3)[C^*(\beta_3,x)+C^*(\beta_2,x)]\Big\}-\mathscr{F}_2P_1\Big\{-\frac{P_3}{2}[C^*(\beta_3,x)\\
&-C^*(\beta_2,x)+S^*(\beta_7,x)+S^*(\beta_9,x)]+P_3S^*(\beta_5,x)+\tfrac{1}{2}(1-4P_3^2+P_3)[S^*(\beta_2,x)\\
&+S^*(\beta_3,x)]\Big\}+V_{10}x \qquad (110)\\
D_2 \approx{}& \mathscr{F}_1\Big\{\tfrac{1}{2}(P_3-4P_1^2P_3)[S^*(\beta_5,x)-S^*(\beta_4,x)]+\frac{P_1^2}{4}[C^*(\beta_8,x)-C^*(\beta_9,x)\\
&-C^*(\beta_6,x)+C^*(\beta_7,x)]+P_1^2P_3[S^*(\beta_7,x)-S^*(\beta_6,x)+S^*(\beta_9,x)-S^*(\beta_8,x)]\Big\}\\
&+\mathscr{F}_1P_3^2z^{-1}\Big\{\tfrac{1}{2}[C^*(\beta_4,x)+C^*(\beta_5,x)]-\tfrac{1}{4}[C^*(\beta_6,x)+C^*(\beta_7,x)+C^*(\beta_8,x)\\
&+C^*(\beta_9,x)]\Big\}-\mathscr{F}_2\Big\{\tfrac{1}{2}(P_3-4P_1^2P_3)[C^*(\beta_4,x)-C^*(\beta_5,x)]+\frac{P_1^2}{4}[S^*(\beta_7,x)\\
&-S^*(\beta_6,x)-S^*(\beta_9,x)+S^*(\beta_8,x)]+P_1^2P_3[C^*(\beta_8,x)-C^*(\beta_9,x)+C^*(\beta_6,x)
\end{aligned}
$$

$$-C^*(\beta_7, x)]\bigg\} - \mathscr{F}_2 P_3^2 z^{-1}\bigg\{\tfrac{1}{2}[S^*(\beta_4, x)+S^*(\beta_5, x)] - \tfrac{1}{4}[S^*(\beta_9, x)+S^*(\beta_8, x)$$

$$+S^*(\beta_7, x)+S^*(\beta_6, x)]\bigg\} + \mathscr{F}_1\bigg\{\tfrac{1}{2}[S^*(\beta_4, x)+S^*(\beta_5, x)] - \frac{P_3}{2}z^{-1}[C^*(\beta_4, x)$$

$$-C^*(\beta_5, x)] + \frac{P_3}{4}z^{-1}[C^*(\beta_8, x)-C^*(\beta_9, x)+C^*(\beta_6, x)-C^*(\beta_7, x)]\bigg\}$$

$$+\mathscr{F}_2\bigg\{\tfrac{1}{2}[C^*(\beta_4, x)+C^*(\beta_5, x)] - \frac{P_3}{2}z^{-1}[S^*(\beta_5, x)-S^*(\beta_4, x)] + \frac{P_3}{4}z^{-1}[S^*(\beta_7, x)$$

$$-S^*(\beta_6, x)+S^*(\beta_9, x)-S^*(\beta_8, x)]\bigg\} + V_{20}x \qquad z \neq 0 \tag{111}$$

$$D_3 = \mathscr{F}_1\bigg\{\frac{P_3}{2}(-1+4P_1^2)[C^*(\beta_4, x)+C^*(\beta_5, x)] - \frac{P_1^2}{4}[S^*(\beta_9, x)+S^*(\beta_8, x)-S^*(\beta_7, x)$$

$$-S^*(\beta_6, x)] - P_1^2 P_3[C^*(\beta_6, x)+C^*(\beta_7, x)+C^*(\beta_8, x)+C^*(\beta_9, x)]\bigg\}$$

$$\mathscr{F}_1 P_3^2 z^{-1}\bigg\{\tfrac{1}{2}[S^*(\beta_5, x)-S^*(\beta_4, x)] - \tfrac{1}{4}[S^*(\beta_7, x)-S^*(\beta_6, x)+S^*(\beta_9, x)$$

$$-S^*(\beta_8, x)]\bigg\} - \mathscr{F}_2\bigg\{\frac{P_3}{2}(-1+4P_1^2)[S^*(\beta_4, x)+S^*(\beta_5, x)] - \frac{P_1^2}{4}[C^*(\beta_6, x)$$

$$+C^*(\beta_7, x)-C^*(\beta_8, x)-C^*(\beta_9, x)] - P_1^2 P_3[S^*(\beta_9, x)+S^*(\beta_8, x)+S^*(\beta_7, x)$$

$$+S^*(\beta_6, x)]\bigg\} - \mathscr{F}_2 P_3^2 z^{-1}\bigg\{\tfrac{1}{2}[C^*(\beta_4, x)-C^*(\beta_5, x)] - \tfrac{1}{4}[C^*(\beta_8, x)-C^*(\beta_9, x)$$

$$+C^*(\beta_6, x)-C^*(\beta_7, x)]\bigg\} + \mathscr{F}_1\bigg\{\frac{P_3}{2}z^{-1}[S^*(\beta_4, x)+S^*(\beta_5, x)] + \tfrac{1}{2}[C^*(\beta_4, x)$$

$$-C^*(\beta_5, x)] - \frac{P_3}{4}z^{-1}[S^*(\beta_9, x)+S^*(\beta_8, x)+S^*(\beta_7, x)+S^*(\beta_6, x)]\bigg\}$$

$$+\mathscr{F}_2\bigg\{\frac{P_3}{2}z^{-1}[C^*(\beta_4, x)+C^*(\beta_5, x)] + \tfrac{1}{2}[S^*(\beta_5, x)-S^*(\beta_4, x)]$$

$$-\frac{P_3}{4}z^{-1}[C^*(\beta_6, x)+C^*(\beta_7, x)+C^*(\beta_8, x)+C^*(\beta_9, x)]\bigg\} + V_{30}x \qquad z \neq 0. \tag{112}$$

where the functions $C^*(\beta_i, x)$ and $S^*(\beta_i, x)$ are defined as

$$C^*(\beta_i, x) \triangleq \int_0^x \int_0^v \cos(2\beta_i u)\,\mathrm{d}u\,\mathrm{d}v$$

$$= \frac{1}{4\beta_i^2}[1-\cos(2\beta_i x)] \qquad (i = 1, \ldots, 9) \tag{113}$$

and

$$S^*(\beta_i, x) \triangleq \int_0^x \int_0^v \sin(2\beta_i u)\, du\, dv$$

$$= -\frac{1}{4\beta_i^2}\sin(2\beta_i x) + \frac{x}{2\beta_i} \qquad (i = 1, \ldots, 9). \tag{114}$$

In summary, given the dimensionless parameters Ω_1, Ω_3, L, z, x, $\mathscr{F}_1$, $\mathscr{F}_2$ and V_{i0} $(i = 1, 2, 3)$, one may proceed as follows to evaluate D_1, D_2 and D_3:

1. determine P_1 and P_3 by reference to equations (73);
2. use equations (81)–(90) to evaluate ζ and β_i $(i = 1, \ldots, 9)$;
3. if $\zeta > 0$, evaluate D_1, D_2 and D_3 by reference to equations (103)–(105) together with the definitions in equations (101) and (102);
4. if $\zeta > 0$, form $\bar{\zeta}$ and $\bar{\beta}_i$ $(i = 1, \ldots, 9)$ by using equations (106), then find D_1, D_2 and D_3 from equations (107)–(109), using the definitions in equations (101) and (102);
5. if $\zeta = 0$, determine D_1, D_2 and D_3 by using equations (110)–(112) together with the definition in equations (113) and (114).

Once again we turn to the computer to obtain a measure of the accuracy of the solution just obtained. In choosing parameter values for this purpose it is well to keep in mind that the approximate solution was generated by using approximate expressions for e_{ij} $(i, j = 1, 2, 3)$, which suggests that the magnitudes of z and x bear directly on the accuracy of D_1, D_2 and D_3; that is, one must expect the discrepancies between the approximate and the exact solution to grow as z and x increase.

Figures 6–10 each show D_1, D_2 and D_3 plotted as a function of x. Numerical solutions of the exact equations of motion are represented by solid curves, while values obtained by using the approximate solution are represented by crosses. In all cases, B^* is presumed to be at rest initially.

Cursory examination of Figs. 6–10 leads to the immediate conclusion that the approximate solution furnishes qualitatively correct results in all cases and that, as expected, quantitative discrepancies increase as x increases. The effect of increasing z can be assessed by comparing Fig. 7 with Fig. 9. As for the effect of other parameters on accuracy, this appears to be negligible; that is, for a given value of z, the accuracy of the approximate solution is independent of the values of Ω_1, Ω_3, $\mathscr{F}_1$, $\mathscr{F}_2$ and L.

Figures 6–10 also permit one to explore the effect of parameter values on the nature of the translational motion. Consider, for example, the parameter L, which characterizes the inertia ellipsoid of the body. Figures 6 and 10 suggest that the mass centers of bodies possessing differently shaped inertia ellipsoids must be expected to follow markedly different trajectories. The force level, on the other hand, plays a relatively minor role as regards the shape of the trajectory, as may be seen by reference to Figs. 7 and 8, which show that, although the values of $\mathscr{F}_1$ and $\mathscr{F}_2$ used to generate Fig. 8 are twice as large as those for Fig. 7, the trajectories followed by the mass center are almost identical. This is not to say, however, that the force level is totally irrelevant to the motion. On the contrary, as shown by equations (103)–(105), (107)–(109) and (110)–(112), D_1, D_2 and D_3 are directly proportional to the magnitude of the applied force $\mathbf{F}$, since $\mathscr{F}_1$ and $\mathscr{F}_2$ are proportional to this quantity; and this proportionality is, in fact, the reason underlying the similarity of Figs. 7 and 8.

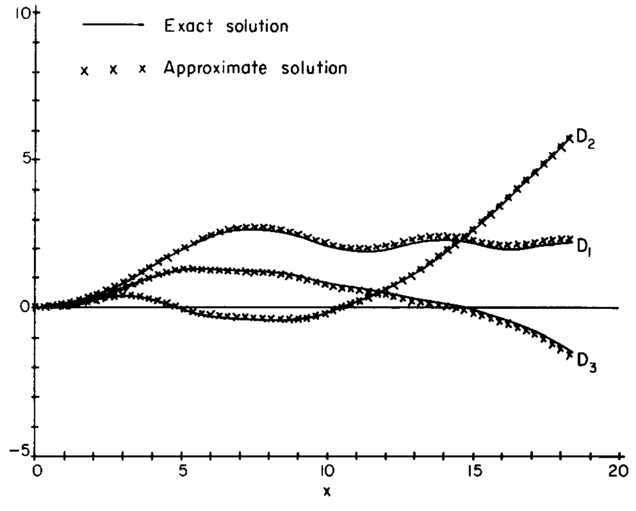

FIG. 6. Comparison of solutions for position coordinates of mass center. $\Omega_1 = 1{\cdot}0$, $\Omega_3 = 0$, $z = 0{\cdot}01$, $L = 9{\cdot}0$, $\mathscr{F}_1 = 0{\cdot}2$, $\mathscr{F}_2 = 0{\cdot}2$, $V_{i0} = 0$ $(i = 1, 2, 3)$.

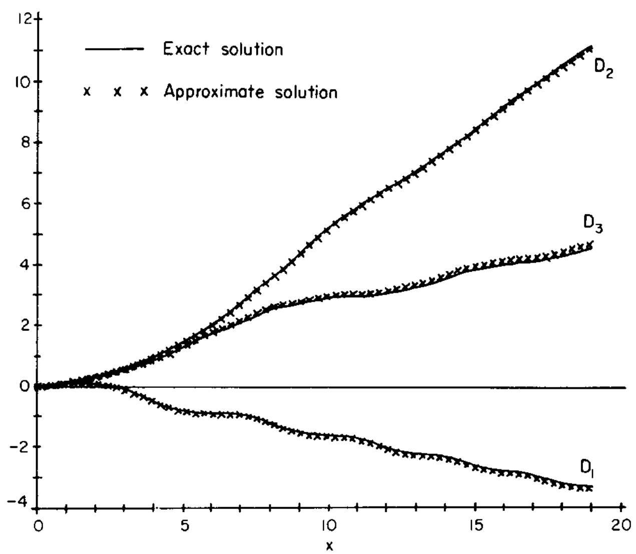

FIG. 7. Comparison of solutions for position coordinates of mass center. $\Omega_1 = 0{\cdot}70711$, $\Omega_3 = 0{\cdot}70711$, $z = 0{\cdot}01$, $L = 9{\cdot}0$, $\mathscr{F}_1 = 0{\cdot}2$, $\mathscr{F}_2 = 0{\cdot}2$, $V_{i0} = 0$ $(i = 1, 2, 3)$.

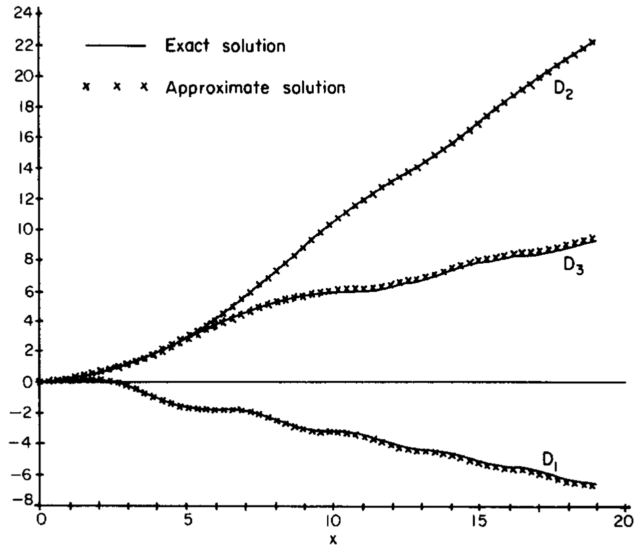

FIG. 8. Comparison of solutions for position coordinates of mass center. $\Omega_1 = 0{\cdot}70711$, $\Omega_3 = 0{\cdot}70711$, $z = 0{\cdot}01$, $L = 9{\cdot}0$, $\mathscr{F}_1 = 0{\cdot}4$, $\mathscr{F}_2 = 0{\cdot}4$, $V_{i0} = 0$ $(i = 1, 2, 3)$.

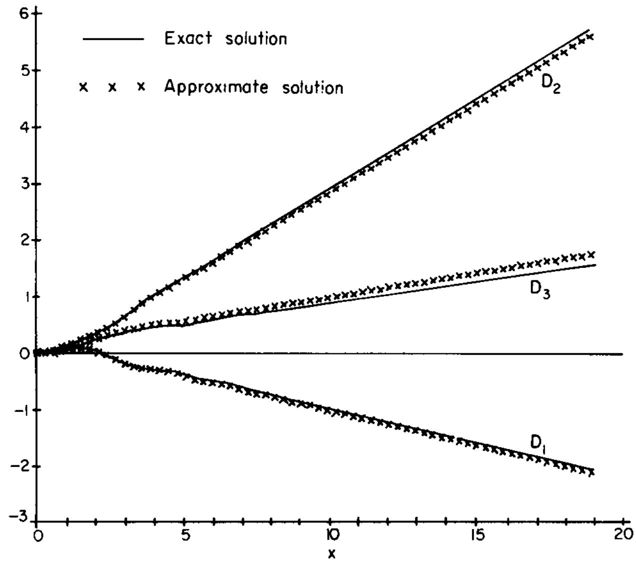

FIG. 9. Comparison of solutions for position coordinates of mass center. $\Omega_1 = 0{\cdot}70711$, $\Omega_3 = 0{\cdot}70711$, $z = 0{\cdot}05$, $L = 9{\cdot}0$, $\mathscr{F}_1 = 0{\cdot}2$, $\mathscr{F}_2 = 0{\cdot}2$, $V_{i0} = 0$ $(i = 1, 2, 3)$.

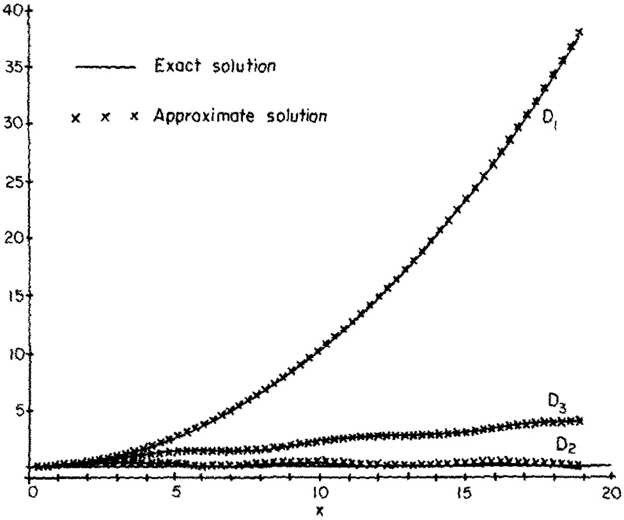

FIG. 10. Comparison of solutions for position coordinates of mass center. $\Omega_1 = 1{\cdot}0$, $\Omega_3 = 0$, $z = 0{\cdot}01$, $L = -0{\cdot}25$, $\mathscr{F}_1 = 0{\cdot}2$, $\mathscr{F}_2 = 0{\cdot}2$, $V_{i0} = 0$ $(i = 1, 2, 3)$.

In conclusion, it can be said that the approximate expressions for D_i $(i = 1, 2, 3)$ describe all essential features of the motion of the mass center and that the agreement between the approximate and exact solution is such as to justify considerable confidence in the former. Furthermore, it may be worth pointing out that this paper applies to a somewhat broader class of problems than that covered by the title. Specifically, the approximate solutions derived here produce good results for certain gyrostats, as will be shown in a subsequent paper.

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Абстракт—Определяются формулы точных динамических и кинематических уравнений, учитывающих движения положения и трансляции, симметрического жесткого тела, под влиянием специальной, связанной с телом, силы. Из этих уравнений, получаются аналитический, но приближенный, запись поведения системы. Сравнивается, затем, важность решения путем сравнения его предсказаний с такими же, вытекающими из решения на ЦВМ точного уравнения движения.